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THE GEOMETRY OF CROSS SECTIONS TO FLOWS†

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In Memory of Ethel Fried

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WE WILL DEVELOP some geometric techniques for finding and classifying cross sections to flows. (The definition of cross section is given at the beginning of §1 below.)

Before discussing our methods, we will summarize two earlier characterizations of cross sections. The first criterion in the literature is due to Schwartzman, who introduced asymptotic cycles of a flow ϕ on a compact metric space [23]. These are real one dimensional homology classes defined for each ϕ-invariant measure. Cross sections to ϕ are determined by an integral one dimensional cohomology class that is positive on all these asymptotic cycles. The essential tools in the proof are the Hahn–Banach Theorem and the Krylov–Bogoliubov theory of invariant measures.

The second criterion in the literature is due to Fuller. Beginning with an angular variable $\theta: M \to S^1$, the function $\theta \circ \phi_t - \theta$ may be regarded as real-valued. If for each $x \in M$ there is a $t > 0$ such that $\theta(\phi_t x) - \theta(x) > 0$, Fuller showed θ is homotopic to a θ' that increases monotonically along the flow. The essential tool in the proof is averaging [11].

These two different approaches both give rise to the cross section as a level curve of a map $\theta: M \to S^1$. Accordingly, the geometry of the cross section itself is hidden, even though the cohomology class determines the homeomorphism type of the manifold [24]. Furthermore, it is clear that the existence of a cross section is in no way dependent on the parameterization of the flow but is a topological property of the underlying foliation.

We will approach the study of cross sections in a direct geometric manner. With few exceptions, we will not use angular variables as a tool. Rather, given an appropriate infinite cyclic covering space $\tilde{M}$ of the manifold M we will lift the flow ϕ on M to a flow $\tilde{\phi}$ on $\tilde{M}$. The orbit space $\tilde{M}/\mathbf{R}$ will furnish a canonical "external cross section" to ϕ. In § 1 below we will show when this external cross section can be imbedded in M as a cross section.

We next examine the various cross sections that exist to a given flow. Using only the oriented foliation, we define "homology directions" in $H_1(M; \mathbf{R})/\mathbf{R}^+$, where this space is topologized as the disjoint union of a unit sphere with the origin. A homology direction is approximately the normalized homology class of a long nearly closed flowline. Similar classes were first considered by Rhodes in the context of asymptotic cycles for maps [22]. We show in § 2 that an integral class in first cohomology determines a cross section if and only if it is positive on all homology directions.

Thus homology directions (of an oriented foliation) serve the same effect as asymptotic cycles (of a flow) for the purpose of finding cross-sections. We compute the homology directions for flows with good symbolic dynamics (such as Axiom A flows) in § 3 and so obtain a finite criterion for the existence of cross sections on such flows.

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Section 4 covers various topics: the behavior of homology directions under perturbation, the topological obstructions on the set of homology directions and computing the homology directions of a suspension flow. It ends with examples that show that minimal sets alone do not suffice to determine the existence of cross sections (although the Birkhoff center is sufficient by § 2).

In § 5 we consider surfaces of section, a more general type of cross section that arises more often in applications. Historically these preceded the notion of cross section and indeed they were first studied by Poincaré. Our criteria of § 1 and § 2 are extended to this wider setting using a geometric device of "blowing up" invariant submanifolds.

We assume throughout this paper that manifolds are C^∞ and all maps and vector fields are C^1. However, our results may clearly be extended to less smooth non-singular flows provided flow box coordinates exist. We also assume, for simplicity, that the local flow ϕ is transverse to ∂M. It would suffice, however, to assume ϕ is transverse to some components of ∂M and tangent to others: one need only double M along the latter set of components and do the various constructions symmetrically.

§1. CROSS SECTIONS AND Z-COVERS

We seek geometric conditions that are necessary and sufficient for a flow ϕ to have a cross section. We will initially assume ϕ is a C^1 flow on a closed C^∞ manifold M.

Recall that a *cross section* K to ϕ is a closed C^1 submanifold of codimension 1 transverse to ϕ that intersects every flowline. We observe that for each point $m \in M$ there is a time $t > 0$ for which $\phi_t m \in K$. This clearly holds when m is uniformly recurrent, i.e. belongs to a minimal set [B1], and it follows for general m by considering a minimal set in the ω-limit set $\omega_\phi(m)$. For $k \in K$ the smallest $t = t(k) > 0$ for which $\phi_t(k) \in K$ is called the *return time* of K. By the Implicit Function Theorem, $t: K \to (0, \infty)$ is a C^1 function. The map $r(k) = \phi_{t(k)}k$, $r: K \to K$, is a diffeomorphism called the *return map* of K for ϕ.

A Z-*cover* is a regular covering space $\pi: \tilde{M} \to M$ with a given identification of its covering group with the integers Z. As is well known, a cross section K to a flow ϕ on M determines a Z-cover $\pi: K \times \mathbf{R} \to M$ by $\pi(k, t) = \phi_t k$. The preferred generator g for the deck transformations is given by $g(r(k), s) = (k, s + t(k))$.

Our analysis of cross sections in this section will begin with a connected Z-cover and determine whether or not it arises from a cross section under the preceding construction. We must, when possible, produce K from $\pi: \tilde{M} \to M$ and ϕ. Our method is to lift ϕ to a flow $\tilde{\phi}$ on $\tilde{M}$ and study the orbit space $\tilde{M}/\mathbf{R}$ of $\tilde{\phi}$. In the situation of the preceding paragraph, $\tilde{\phi}_t(k, s) = (k, s + t)$ on $\tilde{M} = K \times \mathbf{R}$ and $\tilde{M}/\mathbf{R}/ \cong K$. We call $\tilde{M}/\mathbf{R}$ the *external cross section* associated to the flow ϕ and the Z-cover π. Whereas K may be deformed in M, the external cross section is intrinsic. The return map $r: K \to K$ may be recovered as the map induced on $\tilde{M}/\mathbf{R}$ by the covering transformation g^{-1}.

We will prove that the external cross section may be imbedded in M as a cross section provided $\tilde{\phi}$ has the proper geometric behavior. Recall that the Z-cover $\tilde{M}$ has a natural 2-point compactification obtained by adjoining 2 ends $\{-\infty, +\infty\}$, so labeled that $g^i x \to +\infty\ (-\infty)$ as $i \to +\infty\ (-\infty)$ [8]. The required behavior (clearly necessary by the formula for $\tilde{\phi}$ in the last paragraph) is that lifted flowlines go from $-\infty$ to $+\infty$. The original proof, by different methods, is in [11].

THEOREM A. *Let ϕ be a C^1 flow on a compact manifold M. Suppose $\tilde{M}$ is a connected Z-cover of M, with ends $\pm\infty$ as above, and that $\tilde{\phi}$ is the lifted flow.*

Then ϕ admits a connected cross section K which lifts to $\tilde{M} \Leftrightarrow$ for each $x \in \tilde{M}$ we have $\tilde{\phi}_t x \to +\infty$ as $t \to +\infty$ and $\tilde{\phi}_t x \to -\infty$ as $t \to -\infty$.

We will actually prove a generalization of Theorem A needed for later applications [10]. Suppose X is a C^1 vector field on a compact manifold M that is transverse to ∂M. Then X generates a C^1 local flow ϕ. If $I \subset M$ is closed and nonempty and if ϕ induces a flow on I then we call I an *invariant set*. Given ϕ and I, a compact, codimension 1 submanifold $K \subset M$ is a *cross section to ϕ on I* if K is transverse to ϕ, every flowline of $\phi|I$ meets K and $I \cap K \subset \operatorname{int} K$. As in the case $I = M$ discussed previously, K determines a return time and a return map, defined on neighborhoods of $K \cap I$ in K. K also determines a Z-cover of I, which needn't correspond to Z-covers of M (e.g. if I is a null-homologous closed orbit of ϕ). We now obtain a sufficient criterion for the existence of a cross section to ϕ on I.

THEOREM B. *Let M be a compact C^∞ manifold, X a C^1 vector field transverse to ∂M and ϕ the corresponding local flow. Suppose $\pi: \tilde{M} \to M$ is a connected Z-cover, with ends $\pm\infty$ as above, let $\tilde{\phi}$ be the lifted local flow on $\tilde{M}$ and let $\tilde{I} = \pi^{-1} I$.*

There is a cross section K to ϕ on I with associated Z-cover $\pi|\tilde{I} \Leftrightarrow$ for all $x \in \tilde{I}\, \tilde{\phi}_t x \to \infty(-\infty)$ as $t \to +\infty(-\infty)$.

Proof. There is a compact, connected set $G \subset \tilde{M}$ with 2 unbounded complementary regions R_+ and R_-, which are deleted neighborhoods of $+\infty$ and $-\infty$ respectively. We first show that C may be chosen so that each flowline in $\tilde{I}$ meets C in an interval.

For $x \in \tilde{I}$, the flowline $\mathbf{R} \cdot x$ meets C in a compact, nonempty set. We choose (a, b) depending on x so that $(a, b) \cdot x \cap C = \mathbf{R} \cdot x \cap C$. If a codimension 1 transverse disc D centered at x is sufficiently small, then $F(x) = [a, b] \cdot D$ is a flow box with one face $a \cdot D \subset R_-$ and the opposite face $b \cdot D \subset R_+$.

For some finite sequence $x_1, \ldots, x_n \in \tilde{I}$ the sets $\operatorname{int} F(x_i)$ cover $C \cap \tilde{I}$. Let $C' = \bigcup_{i=1}^{n} F(x_i) \cup C$. For $x \in \tilde{I}$, $\mathbf{R} \cdot x \cap C' = \bigcup_{i=1}^{n} (F(x_i) \cap \mathbf{R} \cdot x)$ is a union of closed intervals. If $\mathbf{R} \cdot x \cap C'$ were disconnected, one would have $t_0 x \in b_i \cdot D_i \subset R^+$, $t_1 x \in a_j \cdot D_j \subset R^-$ and $(t_0, t_1) \cdot x$ disjoint from C, which is a contradiction. Thus we may pass from C to C' and assume each flowline meets C in an interval.

The orbit space $\tilde{I}/\mathbf{R}$ is a quotient of $C \cap \tilde{I}$ and hence compact. It is essential to show that $\tilde{I}/\mathbf{R}$ is Hausdorff. Given $y_1, y_2 \in C \cap \tilde{I}$ on distinct flowlines, we must find small closed discs $D(y_i)$ at y_i, transverse to $\tilde{\phi}$ and codimension 1, so that $\mathbf{R} \cdot (D(y_i) \cap \tilde{I})$ are disjoint, $i = 1, 2$. As orbits in $\tilde{I}$ pass through C in intervals of uniformly bounded length, it is easily arranged that $C \cap \mathbf{R} \cdot (D(y_i) \cap \tilde{I})$ be disjoint, $i = 1, 2$: but since C meets every orbit in $\tilde{I}$, this suffices.

We see that the projection $q: \tilde{I} \to \tilde{I}/\mathbf{R}$ is a locally trivial principal $\mathbf{R}$-bundle, with charts of type $((\operatorname{int} D(y)) \cap \tilde{I}) \cdot \mathbf{R}$, with $D(y)$ as in the preceding paragraph. Since the base $\tilde{I}/\mathbf{R}$ is compact and Hausdorff and the fiber $\mathbf{R}$ is contractible, a standard induction argument with partitions of unity gives a section $s: \tilde{I}/\mathbf{R} \to \tilde{I}$. We may choose s so that the image of s lies interior to a compact, codimension 1, transverse submanifold L with $L \cap \tilde{I} = Im(s)$.

We seek to choose L so that it projects into a cross section $K = \pi(L)$ for ϕ on I. We need only choose L to be disjoint from $g^i L$, all $i > 0$, where g is the preferred generator for the deck transformation group. As L is compact, there is an $h = g^{2^n}$ so that $h(L) \cap I \subset (0, \infty) \times h^{-1}L$. We will show that L may be rechosen as L' so that $h(L') \cap \tilde{I} \subset (0, \infty) \cdot L'$. Continuing n times, we will achieve $g(L^{(n)} \cap \tilde{I}) \subset (0, \infty) \cdot L^{(n)}$. We finally take L to be a small neighborhood of $L^{(n)} \cap \tilde{I}$ in $L^{(n)}$ and $K = \pi(L)$.

To construct L' from L, consider the order in which a flowline meets $h^{-1}L$, L, and hL. Let $A = \{y \in \tilde{I}/\mathbf{R} \mid y \text{ meets } L \text{ before } h^{-1}L\}$ and $B = \{y \in \tilde{I}/\mathbf{R} \mid y \text{ meets } L \text{ after } hL\}$. Under the homeomorphism $\hat{h}$ on $\tilde{I}/\mathbf{R}$ induced by h, $\hat{h}A = B$. By hypothesis $\bar{A}$ and $\bar{B}$ are disjoint. Then we define a smooth function $f: L \to [0, \infty)$ so that:

(a) When $q(l)$ is near $\bar{B}$, $f(l) = 0$.

(b) When $q(l) \notin \bar{B}$, $f(l) \cdot l$ lies between $(\mathbf{R} \cdot l \cap h^{-1}L$ and $(\mathbf{R} \cdot l \cap hL$.

Since $\hat{h}A = B$, one easily checks that $L' = \{f(l) \cdot l \mid l \in L\}$ satisfies $h(L' \cap \tilde{I}) \subset (0, \infty) \cdot L'$, as desired. Q.E.D.

Clearly Theorem A follows from Theorem B by choosing $I = M$.

§2. HOMOLOGICAL CRITERIA FOR CROSS SECTIONS

We will determine when a flow ϕ on a closed manifold M has a cross section in terms of certain simple invariants called homology directions. Unlike §1, we will not assume that M is equipped with a preferred $\mathbf{Z}$-cover.

To keep track of the various possible cross sections to ϕ we will use the first integral cohomology group $H^1(M; \mathbf{Z})$. A cross section K determines a class $u_K \in H^1(M; \mathbf{Z})$ since it has a preferred normal orientation given by the flow. If l is an oriented loop transverse to K, then $u_K(l) = \sum_{p \in l \cap K} \epsilon_p$, where $\epsilon_p = +1(-1)$ if the orientation of l agrees (disagrees) with the flow direction at p. This determines (independent of basepoint) a homomorphism $\pi_1 M \to \mathbf{Z}$ which we'll denote u_K as well. When K is connected, the $\mathbf{Z}$-cover $\pi: K \times \mathbf{R} \to M$ considered in §1 is associated to the subgroup $\ker(u_K) \subset \pi_1 M$. The group $H^1(M; \mathbf{Z})$ is free abelian on a finite set of generators. The class u_K arising from a connected cross section K is indivisible (not expressible as mv, $v \in H^1(M; \mathbf{Z})$, $m > 1$) since there is a closed path that meets K transversely in a single point.

A class $v \in H^1(M; \mathbf{Z})$ determines a $\mathbf{Z}$-cover $\tilde{M}$ such that a path δ in $\tilde{M}$ joining x to $g^m x$ ($x \in \tilde{M}$, $m \in \mathbf{Z}$, g the preferred generator for the deck transformations) projects to a closed path $\pi(\delta)$ in M with $u(\pi(\delta)) = m$. This is a natural bijection from $\mathbf{Z}$-covers of M to $H^1(M; \mathbf{Z})$. Here $\tilde{M}$ is connected$\Leftrightarrow u$ is indivisible.

We can now express a uniqueness result that complements the existence result of Theorem A above: cohomology classes determine cross sections up to isotropy.

THEOREM C. *Suppose K and L are cross sections to the flow ϕ on the closed manifold M and that $u_K = u_L \in H^1(M; \mathbf{Z})$. Then there is a smooth family of cross sections K_t with $K_0 = K$, $K_1 = L$.*

Proof. Let ω_K be a closed 1-form on M with $\omega_K(\mathrm{d}\phi/\mathrm{d}t) > 0$, $\omega_K | K = 0$ and $[\omega_K] = u_K$. To construct ω_K, reparameterize ϕ so the return time relative to K is identically one. Then $\pi: K \times \mathbf{R} \to M$, $\pi(k, t) = \phi_t k$ determines the $\mathbf{Z}$-cover corresponding to u_K, it is clear that the form dt on $K \times \mathbf{R}$ is $\mathbf{Z}$-equivariant and induces a form ω_K on M with the desired properties.

Let ω_L be constructed analogously for L and let $\omega_s = (1 - s)\omega_K + s\omega_L$. Then ω_s satisfies

(a) ω_s is nonsingular, $0 \leq s \leq 1$

(b) $[\omega_s] = [\omega_0]$, all s.

Here (a) holds because $\omega_s(\mathrm{d}\phi/\mathrm{d}t) > 0$ and (b) holds because $[\omega_s] =$

$(1-s)u_K + su_L = u_K$. Moser's Lemma shows that (a) and (b) imply that there is an isotopy h_s of M with h_0 the identity and $h_s^*\omega_0 = \omega_s$ [16, 19].

The cross sections $h_s^{-1}(K)$ connect K to some fiber of ω_L which in turn is isotopic to L through cross sections. Q.E.D.

Since isotopic cross sections clearly cannot be distinguished topologically, we will be satisfied to determine which $u \in H^1(M; Z)$ are the classes of cross sections. For indivisible u, Theorem A gave an answer in terms of the connected Z-cover corresponding to u. But if the rank of $H^1(M; Z)$ is more than 1, there are infinitely many connected Z-covers of M and a better approach is needed. Thus we will introduce certain homology invariants that determine which (if any) Z-covers satisfy the hypothesis of Theorem A and so whether ϕ admits any cross section at all.

Suppose M is a compact manifold. Let D_M denote the set $H_1(M; \mathbf{R})/(x \sim rx, r > 0)$ of all directions in the first real homology group of M. The topology given D_M is that of a sphere corresponding to the nonzero homology classes plus an isolated point corresponding to 0. There is a natural projection $p : \pi_1 M \to D_M$.

A sequence $(m_k, t_k) \in M \times (0, \infty)$ is called a *closing sequence based at* m if as $k \to \infty$ one has $m_k \to m$, $\phi_{t_k} m_k \to m$ and $t_k \not\to 0$. Clearly m is the base of a closing sequence $\Leftrightarrow m$ is a nonwandering point for ϕ. Let γ_k be the closed path obtained by joining m to m_k by a short path, following the trajectory $\phi_t m_k$, $0 \le t \le t_k$, and ending with a short path from $\phi_{t_k} m_k$ to m. The sequence $p(\gamma_k) \in D_M$ must have accumulation points, since D_M is compact. Such an accumulation point d is called a *homology direction* for ϕ. When $p(\gamma_k) \to d$ we say (m_k, t_k) is a *closing sequence* for d.

The collection of all homology directions for ϕ is denoted D_ϕ. Clearly D_ϕ is a compact, nonempty set in D_M.

More generally, whenever ϕ is a local flow transverse to ∂M with an invariant set I, one may apply the above procedure to closing sequences in $I \times (0, \infty)$ to define a nonempty compact set $D_{\phi|I}$. In particular, when ϕ has a closed orbit γ, there is a homology direction $d_\gamma = p(\gamma)$ that is the normalized homology class of γ.

A homology direction is always approximable by the normalized homology class of a long, nearly closed trajectory. For it (m_k, t_k) is a closing sequence for d and the t_k's have a bounded subsequence then the trajectory γ through m is periodic and $d = d_\gamma$. But then d has a closing sequence (m, kp), with p the period of γ, with the times $kp \to \infty$.

The relevance of homology directions to cross-sections is seen clearly when ϕ has closed orbit γ. Then γ cuts any cross section K and always in the flow direction so that $u_K(\gamma) > 0$. Note that for classes $d \in D_M$ and $u \in H^1(M; Z)$ one may meaningfully say whether u is positive, zero or negative on d, even though no value $u(d)$ is defined (unless D_M is embedded in $H_1(M; \mathbf{R})$ as the vectors of length 0 and 1 in some norm on $H_1(M; \mathbf{R})$). Thus we say u_K is positive on d_γ. This motivates the following result, the criterion for cross sections we promised earlier.

THEOREM D. *Let ϕ be a flow on a closed manifold M and $u \in H^1(M; Z)$. There is a cross section K to ϕ in class u if and only if u is positive on D_ϕ. Thus ϕ admits a cross section $\Leftrightarrow D_\phi$ lies in an open halfspace of D_M.*

Proof. Suppose u is positive on D_ϕ and indivisible. Then there is a connected Z-cover $\pi : \tilde{M} \to M$ with preferred generator g for the deck transformations. As in §1, there is a 2 point compactification $\tilde{M} \cup \{\pm\infty\}$ for $\tilde{M}$ such that $g^n x \to +\infty(-\infty)$ as $n \to +\infty(-\infty)$ for any $x \in \tilde{M}$. By Theorem A it suffices to show $\phi_t x \to +\infty$ as $t \to +\infty$ (the corresponding property for $-\infty$ follows by considering $\tilde{\phi}_{-t}$, $t \ge 0$).

If x has an ω-limit point $y \in \tilde{M}$ one could pick $r_i \to +\infty$ such that $\tilde{\phi}_{r_i}x \to y$. Then $(\phi_{r_i}\pi x, r_{i+1} - r_i)$ is a closing sequence based at πy. Any homology direction d obtained from this closing sequence would satisfy $u(d) = 0$, a contradiction.

Thus the trajectory $\phi_t m$, $t \geq 0$, can't accumulate in $\tilde{M}$. It must converge to $+\infty$ or $-\infty$, since oscillation between the 2 would produce accumulation points in $\tilde{M}$. If $\phi_t x \to -\infty$ as $t \to +\infty$, one could pick y so that $\pi y \in \omega_\phi(\pi x)$ and then times $r_i \to +\infty$ and integers $n_i \to +\infty$ so that $g^n i(\tilde{\phi}_{r_i x}) \to y$. As above, $(\phi_{r_i}\pi x, r_{i+1} - r_i)$ would be a closing sequence and would produce a homology direction d with $u(d) \leq 0$. This contradiction shows $\phi_t x \to +\infty$, as desired.

Suppose, conversely, that K is a cross section to ϕ, $u = u_k$ and $d \in D_\phi$: we must show u_K is positive on d. Giving $H_1(M; \mathrm{R})$ a norm $\|\cdot\|$, we may identify D_M with vectors of length 0 or 1 in the obvious way. Let (m_k, t_k) be a closing sequence for $d \in D_\phi$ and γ_k the associated sequence of closed paths. Clearly $u_k(\gamma_k) > 0$ for t_k sufficiently large. Thus $[\gamma_k] \neq 0$ and

$$u(d) = \lim_{k \to \infty} u\left(\frac{[\gamma_k]}{\|[\gamma_k]\|}\right).$$

In a compact manifold, a short path cannot represent a large homology class. More precisely, given a Riemannian metric on M, there is a bound over all closed paths α on $(\|[\alpha]\|/\text{length}(\alpha))$[20]. Letting $\alpha = \gamma_k$, and noting that $u(\gamma_k)$ is commensurable with the length of γ_k, we see that $(u(\gamma_i)/\|[\gamma_i]\|)$ is bounded away from 0. Thus $u(d) > 0$, as desired.

Finally, if D_ϕ lies in an open halfspace of D_M there is (by compactness of D_ϕ and the universal coefficient theorem) a $v \in H^1(M; \mathrm{R})$ with $v(D_\phi) > 0$. By passing to a rational approximation, we obtain a class u/n with $n > 0$, $u \in H^1(M; \mathrm{Z})$ and $u/n\,(D_\phi) > 0$. Thus $u(D_\phi) > 0$ and ϕ admits a cross section. *Q.E.D.*

When (as in Theorem B) K is a cross section to $\phi|I, K$ determines a Cech cohomology class $u_K \in H^1(I, \check{Z})$, i.e. ordinary cohomology classes on small neighborhoods of I compatible under restriction maps. If $u \in H^1(M; \mathrm{Z})$ restricts to u_K, we say that K is compatible with u. (Note that the restriction map $H^1(M; \mathrm{Z}) \to \mathrm{H}^1(I; \mathrm{Z})$ isn't generally 1–1 or onto, so one cannot assign a class $u \in H^1(M; \mathrm{Z})$ to K.)

By obvious modification of the above arguments to make use of Theorem B, we obtain

THEOREM E. *Let M be a compact manifold, ϕ a local flow on M transverse to ∂M, I an invariant set and $u \in H^1(M; \mathrm{Z})$. There is a cross section K to $\phi|I$ compatible with $u \Leftrightarrow u$ is positive on $D_{\phi|I}$.*

We now show that the existence of cross sections depends only on the behavior of the flow ϕ on its Birkhoff center C. Recall that C is the largest invariant set for which $C = \Omega(\phi|C)$, where Ω denotes the nonwandering set. In particular $C \subset \Omega(\phi)$, so we'll find that the nonwandering set determines the existence of cross sections.

We begin with the following lemma, in the situation of Theorem E above.

LEMMA 1. *Let Ω denote the nonwandering set of $\phi|I$. There is a cross section L to $\phi|\Omega$ compatible with $u \Leftrightarrow$ there is a cross section K to $\phi|I$ in class u.*

Proof. As one implication is obvious, we suppose L is a cross section to ϕ on Ω and that u_L is the restriction of u to Ω. There is a return map $r: L_1 \to L_2$, where L_1 and L_2 are closed neighborhoods of $L \cap \Omega$ in L, and a return time map $t: L_1 \to (0, \infty)$.

The set $N = \{\phi_g x | x \in L_1, 0 \le t \le t(x)\}$ is a closed neighborhood of Ω.

Any orbit in I spends only a bounded amount of time outside N, since $\alpha(x) \subset \Omega$ and $\omega(x) \subset \Omega$ for all $x \in I$. If $\pi : \tilde{M} \leftarrow M$ is the cover corresponding to U, then it follows that trajectories in $\pi^{-1}(I)$ go from $-\infty$ to $+\infty$. Theorem B now applies.

Q.E.D.

Taking $I = M$ in the preceding lemma and using Theorem E, we obtain

THEOREM F. *If Ω is the nonwandering set of the flow ϕ on the closed manifold M and $u \in H^1(M; \mathbb{Z})$ then there is a cross section K in class $u \Leftrightarrow u$ is positive $D_{\phi|\Omega}$.*

Although Theorem F is sufficient for the application to Axiom A flows in §3, we will sharpen it be replacing Ω by the center C. Again we need a lemma in the setting of Theorem E.

LEMMA 2. *Suppose K is a cross section to $\phi\backslash I$. Then there is a neighborhood N of I such that for any invariant set $J \subset N$, K is a cross section to $\phi\backslash J$.*

Proof. Take N as in Lemma 1.

THEOREM G. *If C is the Birkhoff center of the flow ϕ on the closed manifold M and $u \in H^1(M ; \mathbb{Z})$ and then there is a cross section K in class $u \Leftrightarrow u$ is positive on $D_{\phi|C}$.*

Proof: C is constructed as follows. Let $\Omega_0 = M$ and for each ordinal α for which Ω is defined, let $\Omega_{\alpha+1} = \Omega(\phi|\Omega_\alpha)$. For limit ordinals β, let $\Omega_\beta = \bigcap_{\alpha<\beta} \Omega_\alpha$. As in [2], this sequence is constant from some ordinal δ on and this limiting value is $C = \Omega_\delta$.

Let γ denote the least ordinal such that $\phi|\Omega_\gamma$ has a cross section compatible with u. By Theorem E with $I = C$, we have $\gamma \le \delta$. By Lemma 2, γ cannot be a successor ordinal. Hence, $\gamma = 0$, as desired. *Q.E.D.*

Theorems F and G also follow from [23] since all ϕ-invariant measures are supported on C.

§3. CRITERIA FOR AXIOM A FLOWS AND RELATED RESULTS

We will analyze the previous section's result for an Axiom A flow and find a simpler criterion for the existence of a cross section depending on a Markov partition. This criterion will be extended to other flows with good symbolic dynamics, including pseudo-Anosov flows [7]. For arbitrary flows we obtain an analogous sufficient criterion based on transverse triangulations.

To begin, we recall the suspension construction. Given a homeomorphism $h : X \circlearrowleft$ of a compact metric space X and a continuous function $f : X \to (0, \infty)$, the *mapping torus* M_h is the quotient space of $X \times \mathbb{R}$ by the group generated by $(x, t) \mapsto (h(x), t - f(x))$. The flow $\tilde{\phi}_t(x, s) = (x, s + t)$ induces a flow ϕ_t on M_h called the *suspension flow of* h (with return time f). As the choice of f doesn't affect the conjugacy type of ϕ_t, it will not matter for our purposes. Note that $X \times 0$ determines a cross section to ϕ with return map f.

Recall that a *subshift of finite type* h: $X \circlearrowleft$ is constructed from a finite symbol set F and a directed graph G on F, $G \subset F \times F$, by setting $X = \{(f_i) \in F^{\mathbb{Z}} | (f_i, f_{i+1}) \in G \text{ for all } i\}$ with the product topology and letting h be the shift map on X, $(f_i) \mapsto (f_{i+1})$. We will call the suspension flow ψ of such a h a *flow of finite type.*

Suppose ϕ is a local flow on a compact manifold M transverse to ∂M and $I \subset M$ is an invariant set. If there is a semi-conjugacy s from a flow of finite type ψ onto $\phi|I$ then we say I is *symbolic*. (Recall that s sends oriented trajectories to oriented trajectories.) Clearly closed orbits are symbolic, finite unions of symbolic invariant sets are symbolic and a symbolic invariant set for a subflow is symbolic for the flow.

The most interesting example of a symbolic invariant set is a basic set Λ for an Axiom A flow ϕ. By choosing a Markov family F of local sections for $\phi|\Lambda$ and defining $(f_1, f_2) \in G$ when the rectangle f_1 is stretched across f_2 under a certain return map, Bowen constructed a subshift of finite type and a semiconjugacy s from the suspension flow ψ onto $\phi|\Lambda$[4]. In this case s can be chosen to preserve the time parameter, but we don't require this.

To a symbolic invariant set I, equipped with given F, G and s, we will associate certain closed orbits in I that will determine the existence of cross sections for $\phi|I$.

If a sequence $f_0, \ldots, f_n = f_0$ in F satisfies $(f_i, f_{i+1}) \in G$, $i = 0, \ldots, n-1$, then we call it a *loop* l. Note that l determines a periodic h orbit $\ldots f_{n-1} \cdot f_0 \ldots f_{n-1} f_0 \ldots$ hence a periodic ψ orbit and, applying s, a periodic orbit $\gamma(l)$ in I. If $f_1, \ldots, f_n$ are distinct, we call l *minimal*. Since minimal loops have bounded length, there are only finitely many.

Given a symbolic invariant set I, one may assume by symbol splitting[4] that the cylinder sets $f_0 = \text{constant}$ in $X \times 0 \subset M_h$ have very small images under s. For appropriately small cylinder sets and the corresponding minimal loops, we have

THEOREM H. *Suppose ϕ is a local flow on a compact manifold M transverse to ∂M, I is a symbolic invariant set and $v \in H^1(M; Z)$. There is a cross section to $\phi|I$ in a class $u \Leftrightarrow u(\gamma(l)) > 0$ for all minimal loops.*

Proof. The forward implication is clear, so we suppose u positive on $\gamma(l)$ for all minimal loops l.

Consider a long, nearly closed trajectory $\phi_t m$, $0 \le t \le T_0$, and the corresponding closed loop γ. We must estimate $u(l)$.

Lift this trajectory to a ψ trajectory $\psi_t x$, $0 \le t \le T_1$, $x \in s^{-1}(m)$. Let $f_1, \ldots, f_n$ be the symbols corresponding to the cylinder sets in $X \times 0$ successively visited by this ψ-trajectory. Choose i_1 as large as possible so that $f_{i_1} = f_1$, then i_2 as large as possible so that $f_{i_2} = f_{i_1+1}$, etc. These sequences exhaust $f_1, \ldots, f_N$. One obtains closed orbits $\gamma(\delta_1)$, $\gamma(\delta_2), \ldots, \gamma(\delta_k)$ in I corresponding to the loops $\delta_1 = (f_1 \ldots f_{i_1})$, $\delta f_2 = (f_{i_1+1} \ldots f_{i_2})$, etc. where $k \le \text{card } F$. As γ is obtained by joining $\gamma(\delta_1), \ldots \gamma(\delta_k)$ by a bounded number of bounded curves, up to a small approximation (recall the images of cylinder sets are small), the error $[\gamma] - \sum_{i=1}^{k} [\gamma(\delta_k)]$ is bounded independently of γ.

We now extract from $f_1, \ldots, f_{i_1}$, a minimal loop l, $f_{j_1}, \ldots, f_{j_2} = f_{j_1}$. If ϵ denotes the loop $f_{j_2}, \ldots, f_{i_1}, f_2, \ldots, f_{j_1}$ left over, we have $[\gamma(\delta_1)] = [\gamma(l)] + [\gamma(\epsilon)]$ (again using that cylinder sets have small image). Choosing another minimal loop in ϵ and continuing in this way, we see that $[\gamma(\delta_1)]$ is a positive combination of $[\gamma(l)]$, l minimal, and likewise for the other δ_i.

So within bounded error, $[\gamma]$ is a sum of a large number of $u(\gamma(l))$ (large since $T \to \infty$ implies $N \to \infty$ and minimal loops have bounded length). This shows $(u[\gamma]/\|[\gamma]\|)$ is bounded away from 0 for $T \to \infty$.

If now γ varies over closed paths corresponding to a closing sequence for d, we find

$$u(d) = \lim \frac{u(\gamma)}{\|[\gamma]\|} > 0.$$

Theorem E finishes the proof. Q.E.D.

We globalize Theorem H to obtain the following criterion for cross sections, applicable to Axiom A and pseudo-Anosov flows and the flows whose Ω consists of finitely many basic sets plus finitely many closed orbits.

THEOREM I. *Suppose ϕ is a flow on a closed manifold M and the center C of ϕ is symbolic. Assuming the images of C of cylinder sets are sufficiently small, then for any $u \in H^1(M, \mathbb{Z})$ there is a cross section in class $u \Leftrightarrow u(\gamma(l)) > 0$, where l varies over the finite set of minimal loops.*

Proof. This immediate from Theorems G and H. Q.E.D.

Note that any nonsingular flow ϕ on a closed manifold M is C^0 approximable by Axiom A flows with 1 dimensional Ω [33]. Since a cross section to a flow ϕ is a cross section to all flows C^0 near ϕ, Theorem I gives some information for aribtrary nonsingular flows. In particular, the limit of Axiom A flows without cross section will likewise have no cross section.

Implicit use has been made of the following computation of $D_{\phi|\Lambda}$ when Λ is a basic set for an Axiom A flow.

LEMMA 3. *Let F be a Markov family of small sections to $\phi|\Lambda$ and Y the convex hull of*

$$\{[\gamma(l)] \mid l \text{ a minimal loop for } F\} \subset H_1(M, \mathbb{R}).$$

If $0 \notin Y$ then $D_{\phi|\Lambda} = p(Y)$.

Note. The assumption $0 \notin Y$ is necessitated by the following example: Suppose $F = \{1, 2\}$, $G = F \times F$, $[\gamma(12)]$ is linearly independent of $[\gamma(1)]$ and $[\gamma(1)] + [\gamma(2)] = 0$. Then $0 \notin D_{\phi|\Lambda}$ but $0 \in Y$, so $D_{\phi|\Lambda} \neq p(Y)$.

Proof. The inclusion $D_{\phi|\Lambda} \subset p(Y)$ follows from Theorem H with $I = \Lambda$. So suppose $y \in Y$, that is $y = \sum_{i=1}^{m} a_i \gamma(l_i)$, $a_i \geq 0$, $\Sigma a_i = 1$, we must show $p(y) \in D_\phi$.

We may clearly suppose the a_i rational, since D_ϕ is closed, and, choosing a common denominator $N > 0$, set $a_i = (b_i/N)$, $b_i \in \mathbb{Z}$. Since Λ is a basic set, there is a sequence γ_i in F corresponding to a path through G beginning on l_i and ending on l_{i+1}, i modulo m. Let λ_n denote the loop obtained by concatenating $(nb_1)l_1$'s, γ_1, $(nb_2)l_2$'s, $\gamma_2, \ldots, (nb_m)l_m$'s and γ_m. Clearly $[\gamma(\lambda_n)] = n\Sigma b_i[l_i] + [\gamma(\lambda_0)]$ so $p(\gamma(\lambda_n)) \to p(y)$. This gives $p(y) \in D_\phi$, as desired. Q.E.D.

For an Axiom A–No Cycles flow, it isn't hard to show $D_\phi = \bigcup_\Lambda D_{\phi|\Lambda}$ using the shadowing property. Hence for an Axiom A–No Cycles flow ϕ with cross section, Lemma 3 shows that D_ϕ is a finite complex.

When we take an arbitrary nonsingular flow ϕ of a closed manifold M there are smooth triangulations τ of M such that ϕ is transverse to each $m-1$ simplex, $m = \dim M$ [30]. We call τ a ***transverse triangulation***. To such a τ is associated the finite set F of m simplices and a directed graph G on F, where $(f_1, f_2) \in G$ means that f_1 and f_2 are adjacent and ϕ flows from f_1 into f_2 across their common face. A closed loop l in G determines a closed path $\gamma(l)$ in M up to isotopy (although not necessarily a closed orbit) by the rule that $\gamma(l)$ intersects successively the n simplices listed in l.

THEOREM J. *Let τ be a transverse triangulation for the flow ϕ on the closed manifold*

M and let $u \in H^1(M; Z)$. If for each minimal loop l one has $u(\gamma(l)) > 0$ then there is a cross section to ϕ in class u.

Proof. Take a closing sequence for a homology direction d for d. By perturbation, assume none of the associated closed paths γ_k meet the m–2 skeleton of τ and associate a symbol sequence to γ_k corresponding to the intersections made with $(m-1)$ simplices. Estimating $u(\gamma_k)$ as in Theorem H shows $u(d) > 0$.

Theorem D finishes the argument. Q.E.D.

Simple examples show that there is little constraint on the classes $[\gamma(l)]$ arising from a fine transverse triangulation τ to a flow ϕ with cross section. Consequently there is no strong converse to Theorem J. But if ϕ does admit a cross section, it isn't hard to choose τ so that geodesics joining adjacent faces lie nearly parallel to ϕ. For this τ, the hypothesis of Theorem J will hold, providing a weak converse.

§4. MISCELLANEOUS PROPERTIES OF HOMOLOGY DIRECTIONS

We will make various computations of homology directions in this section and consider the following diverse topics.

(a) Behavior of D_ϕ under perturbation of ϕ.
(b) Construction of ϕ with prescribed D_ϕ.
(c) Computing D_ϕ from a return map.
(d) Cross sections and minimal sets.

Item (a). We will show that when ϕ admits no cross section the set D_ϕ can implode or explode under perturbation of ϕ, but that there is some constraint when ϕ has cross section.

We first consider a parameterized family of translations on tori. Let N be a compact manfold, $g: N \to S^{n-1}$ a smooth map. Define a flow ϕ on $M = N \times T^n$ by $\phi_t(n, x) = (n, x + tg(n))$, where we identify T^n with $\mathbf{R}^n/Z^n$. Identifying S^{n-1} with $D_{T^n} - 0 \subset D_M$ we have

LEMMA 4. $D_\phi = g(N) \subset S^{n-1}$.

We omit the elementary proof. Note this shows D_ϕ can be far more complicated than the finite complexes arising from Axiom A–No Cycles flows with cross section.

Now perturb ϕ by composing with the gradient flow of a Morse-function for N. The new flow ψ has $D\psi = g(\Sigma)$ where Σ is the finite set of singularities for f. For proper choice of g, D_ϕ collapses from S^{n-1} to a point under certain perturbations.

The possibility of D_ϕ exploding was pointed out by F. Wesley Wilson (private communication). One constructs a flow ϕ as in [21] except on $S^2 \times S^1$ rather than S^3. Then the nonwandering set of ϕ is a finite set of null-homologous closed orbits, but certain perturbations ψ of ϕ have closed orbits not homologous to zero.

Now observe ϕ has a cross section K. Then any C^0 near flow ψ also admits K as a cross section. Thus $u(D_\phi) > 0 \Rightarrow u(D_\psi) > 0$, for all $u \in H^1(M; Z)$. This means that if $E \subset D_M$ is a neighborhood of the convex hull of D_ϕ, $D_\psi \subset E$. Thus when ϕ has a cross section the convex hull of D_ϕ cannot exploed under C^0 perturbation of ϕ.

Item (b). For a fixed closed manifold M we will investigate the possible sets D_ϕ for various flows ϕ on M.

There are 2 topological constraints on D_ϕ. First, if $0 \not\in D_\phi$ then ϕ is nonsingular and so $\chi(M) = 0$ [*Hi*]. Second, if D_ϕ lies in an open halfspace of D_M then ϕ admits a cross section K (by Theorem D). This gives a nonsingular closed 1-form ω_K on M with $[\omega_K] = u_K$ (as in the proof of Theorem C). By integration, M fibers over S^1 with fiber K. That has many topological consequences for M. For instance, the Wang exact sequence [28] associated to the fibration

$$H_i(M;\mathbb{R}) \to H_{i-1}(K;\mathbb{R}) \to \mathrm{H}_{i-1}(K;\mathbb{R}) \to H_{i-1}(M;\mathbb{R}) \to \cdots$$

gives (see [17], p. 28 ff) a Morse-type inequality

$$\beta_i - \beta_{i-1} + \cdots \pm \beta_0 \geq 0$$

where $\beta_j = \dim H_j(M;\mathbb{R})$ is the jth Betti number of M. In particular, taking $i = 1$, dim M and dim $M+1$ gives $\beta_1 > 0$ and $\chi(M) = 0$. See also [6, 26, 29].

The following theorem will indicate that there are not further constraints of this type. We call a homology direction $d \in D_M$ ***rational*** if it comes from a rational (or, equivalently, an integral) homology class.

THEOREM K. *Suppose M is a closed manifold, $n = \dim M \geq 3$ and $\chi(M) = 0$. Given any set $S \subset D_M$ which is finite, rational and symmetric (i.e. $S = -S$) there is a flow ϕ on M with $D_\phi = S$.*

Note. It is particularly interesting when $\beta_1 > 0$ and $S = \{\pm d\}$, $d \neq 0$.

We identify $S \subset D_\phi$ with a subset $S' \subset H_1(M;\mathbb{Z})$ such that $p|S'$ is a bijection onto S. We may choose S' so that all closed paths γ with $[\gamma] \in S'$ preserve local orientation.

Proof. Since $\chi(M) = 0$ there is a nonsingular flow ψ on M. We will modify ψ with plugs (see [21, 31]) to obtain a flow ϕ with the properties

(a) $\Omega(\phi)$ consists of finitely many hyperbolic closed orbits $\alpha_1, \ldots, \alpha_n$ with $\{[\alpha_i]\} = S'$.

(b) In a closing sequence for ϕ, the trajectories stay close to one of the α_i.

Clearly (a) and (b) imply $D_\phi = S$.

Given a small positive ϵ, choose fintely many small disjoint discs D_j, $j \in J$, nearly perpendicular to the flowline so that

(1) for each $x \in M$ there is a $t \in (0, \epsilon)$ with $\phi_t \in \bigcup_{j \in J} D_j$

(2) $x \in D_j \Rightarrow \phi_t x \notin D_j$ for $t \in (0, 4\epsilon)$.

Clearly we may take card $J \geq$ card S. Assign to each j a class $s_j \in S'$ so that the map $J \to S'$ is onto.

To each j construct a closed path γ_j so that $[\gamma_j] = s_j$ and $\gamma_j \cap D_j$ is a properly embedded interval in D_j. We may assume γ_j is transverse to ψ and that the $\gamma_j \cup D_j$ are disjoint, for all $j \in J$. Assuming ϵ was chosen small enough, we may pick the γ_j so that

(3) $x \in D_j \cup \gamma_i \Rightarrow \phi_t x \notin D_j \cup \gamma_j$ for $t \in (0, 3\epsilon]$

We extend $D_j \cup \gamma_j$ slightly to an annulus $A_j \cong B^{n-2} \times S^1$ transverse to ψ with core γ_j and with $D_j \cup \gamma_j \subset \operatorname{int} A_j$. Finally we let $R_j = \{\phi_t a_j | a_j \in A_j, 0 \leq t \leq \delta\}$ where δ is very small.

The R_j are disjoint rings diffeomorphic to $R = B^{n-2} \times S^1 \times [0, 1]$. The local flow $\psi|R_j$ is vertical, i.e. preserves the fibers $a \times [0, 1]$. The standard plug [PWW] is a local flow on R, vertical near ∂R, whose local nonwandering set consists of 4 hyperbolic closed orbits in homology class $\pm[S^1]$. We let ϕ be the flow obtained by replacing each $\psi|R_j$ by a standard plug, so chosen that all orbits passing through D_j fall into the sink orbit of the plug and orbits passing through $\psi_\delta D_j$ came from the source orbit of the plug.

It follows easily from the mirror image property of plugs that any long nearly closed trajectory of ϕ is contained in $\bigcup_{j\in J} R_j$. Properties (a) and (b) follow immediately. Q.E.D.

The preceding theorem constructs for each closed M with $\chi(M)=0$, $\beta_1(M)\neq 0$ an Axiom A-finite Ω flow ϕ with no closed orbits homologous to zero. We will now show that for many M a flow with these properties cannot be Morse–Smale. For these M, the flow ϕ cannot be perturbed to a Kupka–Smale flow ψ without blowing up Ω. This gives a simple topological version of the example of [21] in which perturbations of a certain plugged flow on S^3 were analyzed in detail and shown to be non-Morse–Smale. Our examples exist in all dimensions and on manifolds where Morse–Smale flows do occur [1].

THEOREM L. *Suppose M does not fiber over S^1 (or that no such fibration has Morse–Smale monodromy). Then in any Morse–Smale flow ϕ on M there must be a closed orbit γ with $[\gamma]=0\in H_1(M;\mathbf{R})$.*

Proof. By changing ψ near $\Omega=\Omega(\psi)$ one can easily produce a Morse–Smale flow ϕ with $\Omega(\phi)=\Omega$ but such that the flow directions for $\phi|\Omega$ and $\psi|\Omega$ disagree on any prescribed set of orbits in Ω. One sees this most easily if one chooses a round handle decomposition for ψ [1, 18] and observes that the flow on the boundary of a round handle does not determine the orientation of the closed orbit inside.

Suppose all the ψ-orbits γ_i in Ω satisfy $[\gamma_i]\neq 0$. By choosing $\epsilon_i=\pm 1$ properly, the classes $\epsilon_i[\gamma_i]$ will lie in an open 1/2 space of $H_1(M;\mathbf{R})$. Choose ϕ so the orientation of γ_i is reversed if and only if $\epsilon_i=-1$. Then ϕ admits a cross section by Theorem H, and so M fibers over S^1 with Morse–Smale monodromy. Q.E.D.

The discussion preceding Theorem L is summarized in

COROLLARY. *Let M be a closed manifold with $\chi(M)=0$, $\beta_1 M>0$ and such that M is not the mapping torus of a Morse–Smale diffeomorphism. Then M supports a nonsingular Axiom A flow ϕ whose nonwandering set consists of $n<\infty$ closed orbits such that every Kupka–Smale approximation of ϕ has more than n orbits in its nonwandering set.*

Item (c). When ϕ admits a cross section K the homology directions D_ϕ may be calculated in terms of the return map r. Since the suspension construction shows r can be any diffeomorphism whatsoever, this associates a set of homology directions to any diffeomorphism of a compact manifold. We'll see that this generalizes the winding number construction of Poincaré. In some cases, the analogous constructions for asymptotic cycles were developed by Rhodes [22].

The Wang sequence of the fibration $K\to M\to S^1$ ends

$$H_1(K;\mathbf{R})\xrightarrow{r_*-1} H_1(K,\mathbf{R})\longrightarrow H_1(M;\mathbf{R})\xrightarrow{u_K}\mathbf{R}\longrightarrow 0,$$

where we assume, without loss of generality, that K is connected. By Theorem D, $u_K(D\phi)>0$ so that D_ϕ may be identified with a subset of $u_K^{-1}(1)\subset H_1(M,\mathbf{R})$. Translating D_ϕ to $u_K^{-1}(0)$ by an integral vector v and using the Wang isomorphism $u_K^{-1}(0) \cong C = H_1K/im(r_* - 1)$ we obtain a set $D_r \subset C$ called the ***homology directions of r***. As D_r is only defined up to integral translation, one should not regard C as a vector space but as an affine space with an integral lattice.

To compute D_r for some diffeomorphism r of a connected, compact manifold K we choose v as follows. Let $b \in K$ be a base point and δ a path from b to rb inside K. Let ϕ be the suspension flow with return time 1 on the mapping torus $M = M_r$. Following the trajectory $\phi_t b$, $0 \le t \le 1$, by δ^{-1} gives a closed path in M and a class $v \in H_1(M; \mathbb{R})$ with $u_K(v) = 1$.

Suppose $x \in K$ and $r^n x$ is near x and join them by a short path σ. Let ϵ be a path from x to b. Then the path $\epsilon \times \delta \times r\delta \times \cdots \times r^{n-1}\delta \times (r^n\epsilon)^{-1} \times \sigma = \gamma$ is closed. Independent of the choice of ϵ, there is a class $([\gamma]/n) \in C$.

Now every $d \in D_\phi$ admits a closing sequence (x_i, n_i) with $x_i \in K$, $n_i \in \mathbb{Z}$. The associated classes $([\gamma_i]/n_i) \in C$, γ_i as in the preceding paragraph, converge to $(d/u(d)) - v \in D_r$.

This shows how to compute D_ϕ from a return map or, equivalently, how to compute the homology directions of a diffeomorphism r. When r has fixed points, one may choose $b \in \text{Fix}\, f$ and δ the trivial path: one then obtains simpler formulas. If x is also fixed, one obtains the homological part of the Nielsen fixed point class as a homology direction, i.e. $[\epsilon(f\epsilon)^{-1}] \in D_r$ where ϵ joins x to b.

Now suppose ψ is a flow on K and f the time one map. The D_f is obtained as follows. Choose x, n so $f^n x$ is near x. Approximate $\psi_t x$, $0 \le t \le n$, by a closed path. Then $([\gamma]/n) \in C \cong H_1(K, \mathbb{R})$. Taking accumulation points in the usual way gives D_f. This works because there is a natural splitting of the Wang sequence given by the natural homotopy from f to the identity along ψ_t, $0 \le t \le 1$. This case is very much like computating the asymptotic cycles of ψ [R, S1].

In the case of an orientation-preserving diffeomorphism $r: S^1 \circlearrowleft$, it is easy to show $D_r = \{\theta\} \subset \mathbb{R}$, where θ is the rotation number of r. Both D_r and θ are only defined modulo $\mathbb{Z}$, suggesting that it's more natural to study the rotation number in terms of the suspension flow.

We observe that the construction of D_r works just as well for any continuous map $r: K \circlearrowleft$, K a finite complex. There is a corresponding mapping torus M_r and a semiflow ϕ on M_r with cross section K and return map r; one may identify D_r with "homology directions" of ϕ.

Item (d). We will show that minimal trajectories do not suffice to determine the existence of cross sections. This means that one cannot replace the Birkhoff center in Theorem G by $\overline{\bigcup_i X_i}$ where X_i varies over all minimal sets of the flow.

We begin constructing our example with a translational flow ψ on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with generator $\alpha = (a_1, a_2, 1) \in \mathbb{R}^3$ such that every orbit of ψ is dense (this holds for almost all (a_1, a_2)). Choose a function g on T^3 with a nondegenerate minimum of 0 at $0 \in T^3$ with $g(T^3 - 0) > 0$. Let F be the flow generated by the vector field $g\alpha$: one can say F is the flow obtained by stopping ψ quadratically at 0. Let f be the time one map of F. The suspension flow ϕ of f will give our first example.

Note that $T^2 = T^2 \times 0 \subset T^3$ is almost a cross section to ϕ, so that there is a return time map $\tau : T^2 - 0 \to (0, \infty)$. By solving $(\dot{x}, \dot{y}) = (x^2 + y^2, 0)$ for $|x| \le 1$ and $|y|$ small, one sees that τ/σ is bounded away from 0 and ∞, where $\sigma(x) = (1/\text{dist}\,(x, 0))$ for $x \in T^2 - 0$. Integrating with respect to Lebesgue measure shows $\int \sigma < \infty$ so that $\int \tau = C < \infty$ as well. We will show

LEMMA 4. $\alpha/C \in D_f$.

Proof. Note that $0 \in \omega_\psi(x)$ for all $x \in T^3$. It follows that $0 \in \omega_F(x)$ for all $x \in T^3$. As ϕ goes very slowly near 0, we see $0 \in \omega_f(x)$ as well.

Let $\beta = (a_1, a_2) \in \mathbb{R}^2$. The translation by β gives a minimal transformation T_β of T^2 (T_β is the return map of a cross section to the minimal flow ψ). Such T_β are ergodic and the Birkhoff Ergodic Theorem gives

$$\frac{1}{N}\sum_{i=0}^{N-1} \tau T^i x \to C, \text{ for almost all } x.$$

Choose such an x near 0. By the preceding paragraph, there is a large m with $f^m x$ near 0. We suppose that $F_{t_0}x \in T^2$ for some $t_0 \in [m, m+1]$. Then if γ is obtained by closing $F_t x$, $0 \le t \le m$, by a short path, we find $1/m\,[\gamma] \cong N/m\,\alpha$, where $\sum_{i=0}^{N-1} \tau T_\beta{}^i = t_0$. Hence $m/N \cong C$, so $1/m\,[\gamma] \cong \alpha/C$. By the results of part C of this section, $\alpha/C \in D_f$.

Q.E.D.

We see from the above proof that ϕ has a unique minimal set γ corresponding to the fixed point $0 \in T^3$. But the lemma shows $D_\phi \neq \{p(\gamma)\}$. Hence there are cohomology classes u with $u(\gamma) > 0$ that don't arise from cross sections.

We can improve this example to obtain a flow ϕ on T^5 where minimal sets are finitely many closed orbits, all in the same nonzero homology class, that has no cross section at all. Observe that by multiplying g by a constant, C can take any positive value. Varying α as well, we can make α/C approximate any desired vector in $H_1(T^3; \mathbb{R})$.

In terms of the mapping torus $M_f \cong T^4$, with its natural cohomology class u, we can pick ϕ so that D_ϕ contains $[\gamma]$ and d, where d is closed to any prescribed direction δ with $u(\delta) > 0$.

We now fix a class $h \in H_1(T^4 : \mathbb{Z})$. Every u with $u(h) = 1$, $u \in H^1(T^4; \mathbb{Z})$, determines a product structure $T^4 = T^3 \times S^1$ with $[S^1] = h$. By varying u as well as α and C, we can produce a flow ϕ on T^4 whose unique minimal set is a closed orbit γ with $[\gamma] = h$ and a class $d \in D_\phi$ near any prescribed nonzero direction in D_{T^4}. Choose $\phi_1, \ldots, \phi_4$ in this way so that 0 lies in the interior of the convex hull of $\{h, d_1, \ldots, d_4\}$ in $H_1(T^4; \mathbb{Z})$. Define Φ to agree with ϕ_i on $T^4 \times p_i$, where $p_1, \ldots, p_4$ are distinct points in S^1, and to wander elsewhere. It's clear from Theorem D that Φ has the properties mentioned above.

It would be interesting to know whether minimal sets determine the existence of cross sections in dimension 3, or even on the solid torus [12].

§5. SURFACES OF SECTION

The results obtained above for cross section to flows may be usefully extended to criteria for the existence and classification of Birkhoff's "surfaces of section." Recall that given a flow on a manifold M, a codimension one closed submanifold $K^{n-1} \subset M^n$ is a *surface of section* bounded by J^{n-2} for ϕ if

(1) J is a union of component of ∂K, each invariant under ϕ,

(2) $d\phi/dt$ is transverse to $K - J$,

(3) the angle between $d\phi/dt$ and K does not vanish to first order as one approaches a point in J and

(4) every flowline of ϕ intersects K in uniformly bounded time.

This definition is given in Birkhoff [2, 3] where surfaces of section were shown to exist for many Hamiltonian systems with two degrees of freedom. This work generalized Poincaré's construction of an annular surface of section in the restricted three-body problem, which was the first use of this method for studying dynamical systems. Note that if J is empty, one has simply the definition of a cross section to ϕ. Also, a surface of section K determines a well-defined first return map whose dynamics closely reflect those of ϕ.

To better visualize surfaces of section, it is useful to consider an auxiliary manifold M^* and flow ϕ^* so that certain cross sections to ϕ^* correspond to surfaces of section of ϕ bounded by J. A closed submanifold $J^i \subset \operatorname{int} M^m$, where $i < m$ and $\partial J = \emptyset$, gives rise to a manifold M^* by deleting from M an open tubular neighborhood N of J. As in knot theory, M^* will be called the *exterior* of J. We will denote the boundary $\bar{N} - N \subset M^*$. By the uniqueness of tubular neighborhoods up to isotopy [13], M^* may be given a well-defined differentiable structure.

For present purposes it is more natural to view M^* as obtained by replacing points $j \in J$ by their sphere of nonzero normal directions $[(T_jM/T_jJ) - \{0\}]/\mathbb{R}^+$. From this viewpoint one constructs an atlas for M with charts $\{(U_i, f_i), (V_j, g_j)\}$, where

(1) $U_i \cap J = \emptyset$.

(2) $g_j : V_j \to R^j \times r^{n-j}$ represents V_j as an R^{n-j} bundle over $V_j \cap J \cong R^j \times 0$.

The smooth transition functions $\tau_{jj'} : g_j(V_j \cap V_{j'}) \to g_{j'}(V_j \cap V_{j'})$ preserve the points in $R^j \times 0$. By passing to spherical coordinates, one obtains a map

$$\tau^*_{jj'} : g_i(V_j \cap V_{j'} \cap J) \times [0, \infty) + S^{m-j-1} \to$$

$$g_{j'}(V_j \cap V_{j'} \cap J) \times [0, \infty) \times S^{m-j-1}$$

that induces $\tau_{jj'}$ when one identifies (p, r, v) to (p, rv). The points $(p, o. v)$ are naturally interpreted as the nonzero normal directions to J at P. This gives an atlas $\{U_i, f_i), (V_j^*, g_j^*)\}$ on M^* where $g_j^*(V_j) = g_i(V_j \cap J) \times [0,\infty) \times S^{m-j-1}$ and $\tau^*_{jj'} g_g^* = g^*_{j'}$. Since one uses the derivative of $\tau_{jj'}$ in the directions normal to J and define $\tau^*_{jj'}$, the natural differentiability class of M^* is C^{r-1} for a C^r pair (M, J).

Given a flow ϕ on M and an invariant closed submanifold J there is a naturally defined flow ϕ^* on M defined by the action of ϕ on the points $M - J$ and on the nonzero normal directions to J. We will call ϕ^* the flow induced by ϕ on the exterior of J or the *blown up flow*.

For the case of interest for surfaces of section, one has $j = n - 2$ and the vector field $(\mathrm{d}\phi^*/\mathrm{d}t)$ is determined locally by $(\mathrm{d}\phi/\mathrm{d}t)$ as follows. By means of a chart (V, g), $(\mathrm{d}\phi/\mathrm{d}t)$ may be regarded as a vector field on $R^{n-2} \times R^2$. Thus $(\mathrm{d}\phi/\mathrm{d}t)\ (j, x, y) = (\mathrm{d}j/\mathrm{d}t, \mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t)\ (j, x, y)$ where $(\mathrm{d}x/\mathrm{d}t) = (\mathrm{d}y/\mathrm{d}t) = 0$ at points $(j, 0, 0)$. One may write

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}(j, x, y) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x}{\mathrm{d}t}(j, sx, sy)\right)\mathrm{d}s \\ &= x\int_0^1 \frac{\partial}{\partial x}\left(\frac{\mathrm{d}x}{\mathrm{d}t}(j, sx, sy)\right)\mathrm{d}s \\ &\quad + y\int_0^1 \frac{\partial}{\partial y}\left(\frac{\mathrm{d}x}{\mathrm{d}t}(j, sx, sy)\right)\mathrm{d}s \\ &= ax + by\end{aligned}$$

and likewise $(\mathrm{d}y/\mathrm{d}t)\ (j, x, y) = cx + dy$ where a, b, c and d are continuous functions of (j, x, y). In cylindrical coordinates (j, r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$, one has

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\cos\theta + \frac{\mathrm{d}y}{\mathrm{d}t}\sin\theta$$

$$= (ax + by)\cos\theta + (cx + dy)\sin\theta$$

$$= r(a\cos^2\theta + (b+c)\cos\theta\sin\theta + d\sin^2\theta)$$

$$\frac{d\theta}{dt} = \frac{1}{r}\left(\frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta\right)$$

$$c\cos^2\theta + (d-a)\cos\theta\sin\theta - b\sin^2\theta)$$

Hence one defines

$$\frac{d\phi^*}{dt}(j, r, \theta) = \left(\frac{dj}{dt}(j, r\cos\theta, r\sin\theta), r(a\cos^2\theta +\right.$$

$$(b+c)\cos\theta\sin\theta + d\sin^2\theta), c(\cos^2\theta + (d-a)\cos\theta\sin\theta - b\sin^2\theta)$$

where $j \in J$, $r \in [0, \infty)$ and $\theta \in R/2\pi Z$.

By investigating the behavior of this expression under smooth changes of variables $\tau(j, x, y) = (j', x', y')$ that preserves $R^{n-2} \times (0, 0)$ one may readily show the blown up flow ϕ^* is well-defined. Note also that $(d\phi^*/dt)$ $(j, 0, \theta) = (dj/dt\,(j, 0, 0),), c(j, 0, 0)\cos^2\theta + (d-a)\cos^2\theta - b\sin^2\theta)$, so ϕ^* preserves the collection of circles $j = j_0$, $r = 0$. As these circles are the normal directions over the point j_0, it is easy to see that one has a semiconjugacy $\pi: M^* \to M$, $\phi_t\pi = \pi\phi_t^*$, where $\pi|M^* - J = id$ and $\pi: J^* \to J$ is the fibration of the unit normal bundle. The flow $\phi^*|J^*$ may be obtained invariantly by passing from $(d\phi/dt)$ to the quotient flow on $V = T_JM/TJ$ to the further quotient flow on $(V - O \text{ section})/\mathbb{R}^+ \cong J^*$. This shows that the flow ϕ^* is independent of the choice of the disc bundle over J used to define the charts (V_j, g_j).

Now that the flow ϕ^* is available, one may recast the study of surfaces of section of ϕ bounded by J into problems about cross sections to ϕ^*. The following lemma summarizes this correspondence—its proof is elementary and will be omitted.

LEMMA 5. *A surface of section (K, J) for a flow ϕ on M determines a cross section K^* to ϕ^* such that*

(1) *∂K^* intersects each fiber of $\pi: J^* \to J$ transversely in a single point and*

(2) *π maps K^* diffeomorphically onto K.*

Conversely, if K^* is a cross section to ϕ^* which intersects each fiber $\pi^{-1}(j)$ transversely and exactly once then $\pi(K^*)$ is a surface of section for ϕ bounded by J.

As noted by Birkhoff [2], the existence of a nonempty, codimension two invariant submanifold is a serious restriction on ϕ. If, however, one is given such a submanifold J, one may determine whether J bounds a surface of section K for ϕ. If K exists then $K^* \cap J^*$ gives a section of the unit normal bundle to J and the transverse flow ϕ^* gives an orientation to each fiber in the circle bundle $J^* \to J$. Thus, the normal bundle of J must be trivial in order for J to bound a surface of section.

THEOREM M. *Let J be a codimension two submanifold of M invariant under the flow ϕ with trivial normal bundle. Let $u \in H^1(M^*; Z)$ be indivisible. Then there is a surface of Section K for ϕ bounded by J with K^* representing u if and only if*

(1) *for each fiber F of $\pi: J^* \to J$ one has $u[F] = \pm 1$, and*

(2) *there is a cross section to ϕ^* in class u.*

Proof. The necessity of conditions (1) and (2) is clear from the last proposition. So, suppose that (1) and (2) hold.

The trivialization of $J \times S^1 \cong J^*$ of the circle bundle $J^* \to J$ may be modified by any diffeomorphism of $J \times S^1$ which maps each fiber into itself. Using property (1) one may assume the trivialization so chosen that $u|J^* = p^*(1)$ where p is the projection $p : J^* = J \times S^1 \times S^1$. As the cohomology class of $p : J^* \to S^1$ is the restriction of a class on M^*, p extends to a map $p : M^* \to S^1$ with $p^*(1) = u$.

Adopting the averaging argument of [11], one may lift p to the infinite cyclic cover $\tilde{M}^*$ determined by u so that $p : J \times S^1 \to S^1$ lifts to the projection map $\tilde{p} : \tilde{J}^* = J \times R \to R$. Then define $\tilde{p}_T(x) = (1/T) \int_0^T \tilde{p}(\tilde{\phi}_t x)\, \mathrm{d}t$. Since there is a cross section to $\phi^*|J^*$ in class $u|J^* = P^*(1)$, the values of $\tilde{p}(\tilde{\phi}_T x) - \tilde{p}(x)$ tend uniformly to $+\infty$ as $T \to +\infty$ *and* in fact satisfy $\tilde{p}(\tilde{\phi}_T x) - \tilde{p}(x) > C \cdot T + D$, $C > 0$, $d \in R$. It follows that

$$\tilde{p}_T(\tilde{\phi}(x)) - \tilde{p}_T(x) = \frac{1}{T} \int_0^{\epsilon} \tilde{p}(\tilde{\phi}_{TS} x) - \tilde{p}(\tilde{\phi}_X s)\, \mathrm{d}s$$

$$\geq \frac{\epsilon}{T}(CT + D) > \frac{1}{2} C\epsilon$$

for large T. Thus $D\tilde{p}_T(x)\ (\mathrm{d}\tilde{\phi}/\mathrm{d}t) \geq C/2 > 0$ for large T.

Also $\tilde{p}_T$ is Z-equivariant, that is $\tilde{p}_T(j, \theta + 2\pi) = \tilde{p}_T(j, \theta) + 2\pi$. Hence $\tilde{p}_T$ induces a map $p_T : J^* \to S^1$. Clearly $p_T{}^*(1) = p_0{}^*(1) = p^*(1) = u$ and p_T is a fibration transverse to $(\mathrm{d}\phi/\mathrm{d}t)$. Hence if K^* is a fiber of p_T, $K^* = p_T{}^{-1}$ (point), then K^* is a cross section to ϕ^*.

But notice that with the fixed trivialization $J \times \mathbf{R} \cong J^*$ one has $D\tilde{p}_T(x)\ (0, 1) = 1/T \int_0^T D\tilde{p}(_S x)\ (0, 1)\, \mathrm{d}s \geq c_0$ for some positive constant c_0 and any $x \in J^*$. This is because the flow $\tilde{\phi}|J^*$ covers the flow $\phi|J$ and each fiber $j \times R$ is mapped diffeomorphically and in an order preserving way by each map $\tilde{\phi}_s$, $0 \leq s \leq T$. Thus, one obtains that $K^* \cap J^*$ is transverse to the fibers of $J^* \to S^1$, as desired.

To generalize Theorem D to surfaces of section one must extend the idea of homology directions and study the cohomology of the exterior manifold M^* considered above. For this it is simpler to regard M^* as the complement of a tubular neighborhood of J.

LEMMA 6. *If J is a closed, codimension 2 submanifold of M with trivial normal bundle then there is a short exact sequence $0 \to H^1(M; Z) \to H^1(M^*; Z) \to H^0(J; Z) \to 0$ which may be split to give $H^1(M^*; Z) \simeq H^1(M; Z) \oplus H^0(J; Z)$.*

Proof. Writing a tubular neighborhood of J in M as $J \times D^2$, inclusion of pairs induces a map α of the cohomology of a long exact sequence of $(M, J \times D^2)$ to the sequence of $(M^*, J \times S^1)$. By excision α is an isomorphism of $H^*(M, J \times D^2) \to H^*(M^*, J \times S^1)$. By the Kunneth formula α is one-to-one on $H^*(J \times D^2) \to H^*(J \times S^1)$. So by the 5 Lemma, all the maps are injective. Hence the quotient complex coker (α) is exact, as may be easily seen by the Snake Lemma. Since coker $(\alpha : H^1(J \times D^2) \to H^1(J \times S^1))$ is zero, one obtains

$$\operatorname{coker}(\alpha : H^1 M \to H^1 M^*) \cong \operatorname{coker}(\alpha : H^1(J \times D^2) \to H^1(J \times S^1))$$

$$\cong \operatorname{coker}(H_1{}^* : H^1 J \to H^1(J \times S^1)) \simeq H^0 J.$$

Hence the sequence $0 \to H^1 M \to H^1 M^* \to H^0 J \to 0$ is exact. As $H^0 J$ is free the sequence splits as desired.

By dualizing this argument, one obtains $H_1(M^*; Z) \cong H_1(M; Z) \oplus H_0(J; Z)$.

Modding out torsion gives $H_1(M^*; Z)/\text{torsion} \simeq H \oplus H_0(J; Z)$ where $H = H_1(M; Z)/\text{torsion}$. This last isomorphism may be interpreted geometrically as follows. One chooses loops γ_i so that $[\gamma_i]$ are an integral basis for H. By perturbation one may assume $\gamma_i \subset M^*$. For a loop β in M^* with $[\beta] = \sum_i \alpha_i [\gamma_i] \in H$, one chooses a 2-chain c with $\partial c = \beta - \Sigma \alpha_i \gamma_i$ (modulo torsion).

Counting each component of J according to its algebraic intersection with c gives a class $\omega(\beta) \in H_0(J; Z)$ which is the winding number in case $H = 0$ and J is connected. The above isomorphism carries $[\beta] \in H_1(M^*)/\text{torsion}$ to $(\Sigma_{\alpha i} [\gamma_i], \omega(\beta))$.

Hence the sequences used to define the homology directions $d \in D_{\phi *}$ can be understood geometrically on the manifold M, without actually constructing M^*. Given an invariant codimension 2 submanifold with trivialized normal bundle, define $D_{\phi, J} \subset H_1(M; R) \oplus H_0(J; R)$ by the following process. Suppose that

(1) $m_i \in M - J$, $m_i \to m \in M$

(2) $t_i \not\to +0$

(3) $\phi_{t_i} m_i \to m$

(4) if $m \in J$, the sequences m_i and $\phi_{t_i} m_i$ approach m along a certain normal direction v at m

(5) $\pi([\gamma_i], \omega[\gamma_i]) \to d \in D = \{0\} \cup$ (unit sphere in $H_1M \oplus H_0J$) where γ_i is the loop obtained by closing the path $\phi_t m_i$, $0 \le t \le t_i$, by a short geodesic and $\pi : H_1(M: Z) \oplus H_0(J; Z) \to D$ is given by normalization by positive scalars. Then $d \in D_{\phi, J} =$ *the homology directions of* ϕ *relative to* J.

It is easy to see that $D_{\phi *}$ corresponds to $D_{\phi, J}$ under the isomorphism $H_1M^* \cong H_1M \oplus H_0J$. Thus we may summarize the above results in

THEOREM N. *There is a surface of section for ϕ with boundary J corresponding to $u \in H^1M \oplus H^0J$ if and only if*

(1) *u takes value ± 1 on each standard generator of H_0J and*

(2) $u(D_{\phi, J}) > 0$.

While it is certainly possible that a flow ϕ has no surface of section for any choice of J, this is much more likely than the existence of a cross section (corresponding to $J = \emptyset$). For instance, flows with surfaces of section are easily constructed on the n sphere, with $J = (n-2)$ sphere. Indeed, it is easy to see that a manifold M admits a flow with a surface of section if and only if M has an open book decomposition[32], which is far weaker than the condition that M fibier over S^1. It follows from [15] that flows with surface of section exist on any odd dimensional manifold M.

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